

Physics 110B
Homework Solution #3

#1 (Griffiths 10.3)

$$\bar{E} = -\bar{\nabla}V - \frac{\partial \bar{A}}{\partial t} = \boxed{\frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}} \quad \bar{B} = \bar{\nabla} \times \bar{A} = \boxed{0}$$

\bar{A} is a vector in the radial direction, therefore it has no curl.

#2 (Griffiths 10.5)

$$(\text{eq 10.7}) \quad V' = V - \frac{\partial \lambda}{\partial t} = -\left(-\frac{1}{4\pi\epsilon_0} \frac{q}{r}\right) = \boxed{\frac{1}{4\pi\epsilon_0} \frac{q}{r}}$$

$$(\text{eq 10.7}) \quad \bar{A}' = \bar{A} + \bar{\nabla}\lambda = \cancel{-\frac{1}{4\pi\epsilon_0} \frac{qt}{r^2} \hat{r}} + \cancel{\left(-\frac{1}{4\pi\epsilon_0} \frac{qt}{r}\right)\left(-\frac{1}{r^2} \hat{r}\right)} = \boxed{0}$$

This gauge function transforms the above nonstandard potentials to the standard potentials that we are used to.

#3 (Griffiths 10.7)

Let's begin with a divergence of \bar{A} that does not satisfy the Lorentz gauge, and then we'll show that we are always able to find a gauge transformation which allows us to satisfy the Lorentz gauge:

$$\bar{\nabla} \cdot \bar{A} \neq -M_0\epsilon_0 \frac{\partial V}{\partial t} \quad \Rightarrow \quad \bar{\nabla} \cdot \bar{A} + M_0\epsilon_0 \frac{\partial V}{\partial t} = B \quad \text{a known fnc}$$

Pick now \bar{A}' and V' to satisfy the Lorentz gauge,

$$\bar{\nabla} \cdot \bar{A}' = -M_0\epsilon_0 \frac{\partial V'}{\partial t}$$

Thus,

$$\bar{\nabla} \cdot \bar{A}' + \mu_0 \epsilon_0 \frac{\partial V'}{\partial t} = \bar{\nabla} \cdot \bar{A} + \nabla^2 \lambda + \mu_0 \epsilon_0 \frac{\partial V}{\partial t} - \mu_0 \epsilon_0 \frac{\partial^2 \lambda}{\partial t^2}$$

using (eq 10.7)

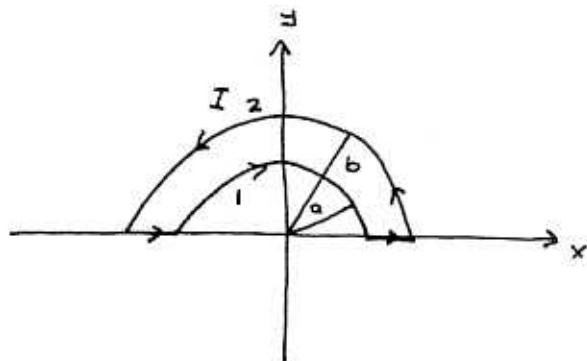
$$= \left(\bar{\nabla} \cdot \bar{A} + \mu_0 \epsilon_0 \frac{\partial V}{\partial t} \right) + \left(\nabla^2 \lambda - \mu_0 \epsilon_0 \frac{\partial^2 \lambda}{\partial t^2} \right) = \bar{B} + \nabla^2 \lambda$$

\Rightarrow This will equal zero, and thus the Lorentz gauge, if we pick for λ the solution to $\boxed{\nabla^2 \lambda = -\bar{B}}$, which we in fact know how to do.

- We can always find a gauge in which $V' = 0$, by picking $\boxed{\lambda = \int_0^t V dt'}$
- We cannot in general pick $\bar{A} = 0$ since this would make $\bar{B} = 0$

#4 (Griffiths 10.10)

$$\bar{A} = \frac{\mu_0}{4\pi} \int \frac{I(t_r)}{r} d\bar{l} \quad (\text{eq 10.19})$$



$$= \frac{\mu_0 k}{4\pi} \int \frac{(t - t'/c)}{r} d\bar{l}$$

$$= \frac{\mu_0 k}{4\pi} \left(t \oint \frac{d\bar{l}}{r} - \frac{1}{c} \oint d\bar{l}' \right) = \frac{\mu_0 k t}{4\pi} \left(\frac{1}{a} \int_1 d\bar{l} + \frac{1}{b} \int_{I_2} d\bar{l} + 2\hat{x} \int_a^b \frac{dx}{x} \right)$$

$$= \frac{\mu_0 k t}{4\pi} \left(\frac{1}{a} 2a + \frac{1}{b} (-2b) + 2 \ln \frac{b}{a} \right) \hat{x}$$

$$\boxed{\bar{A} = \frac{\mu_0 k t}{2\pi c} \ln \frac{b}{a} \hat{x}}$$

The changing magnetic field induces the electric field.

$$\bar{E} = -\frac{\partial \bar{A}}{\partial t} = \boxed{-\frac{\mu_0 k}{2\pi c} \ln \frac{b}{a} \hat{x}}$$

Since we only know \bar{A} at one point, the center we cannot compute $\bar{\nabla} \times \bar{A}$ to get \bar{B}

(3)

#5 (Griffiths 10.13)

$$\bar{w}(t) = (a \cos \omega t, a \sin \omega t)$$

$$\bar{v}(t) = (-a \omega \sin \omega t, a \omega \cos \omega t)$$

$$\bar{r} = \bar{r} - \bar{w}(t)$$

$$= z \hat{z} - (a \cos \omega t \hat{x} + a \sin \omega t \hat{y})$$

$$r^2 = z^2 + a^2 \cos^2 \omega t + a^2 \sin^2 \omega t = z^2 + a^2$$

$$\Rightarrow r = \sqrt{z^2 + a^2}$$

$$\bar{r} \cdot \bar{v} = -a^2 \omega (-\sin \omega t, \cos \omega t, +\sin \omega t, \cos \omega t) = 0$$

Thus,

$$V(\bar{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{q_c}{(r_c - \bar{r} \cdot \bar{v})} = \frac{1}{4\pi\epsilon_0} \frac{q}{r} = \boxed{\frac{1}{4\pi\epsilon_0} \frac{q}{\sqrt{z^2 + a^2}}} \quad (\text{eq 10.39})$$

$$(\text{eq 10.40}) \quad \bar{A}(\bar{r}, t) = \frac{\bar{v}}{c^2} V(\bar{r}, t) = \boxed{\frac{1}{4\pi\epsilon_0 c^2} \frac{q a \omega}{\sqrt{z^2 + a^2}} (-\sin(\omega t_r) \hat{x} + \cos(\omega t_r) \hat{y})}$$

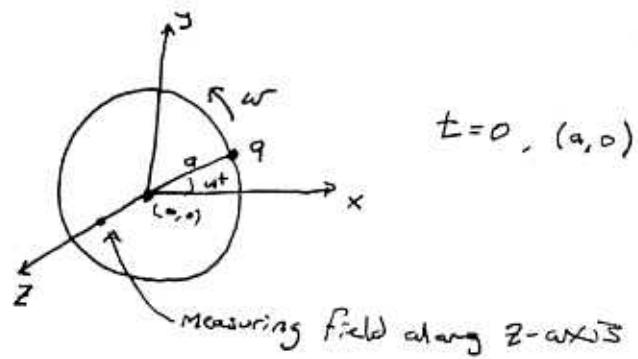
where, $t_r = t - \frac{\sqrt{z^2 + a^2}}{c}$

#6 (Griffiths 10.14)

$$(\text{eq 10.42}) \quad V(\bar{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{q_c}{\sqrt{(c^2 t - \bar{r} \cdot \bar{v})^2 + (c^2 - v^2)(r^2 - c^2 t^2)}}$$

Evaluating the term under the square root we have,

$$S = c^4 t^2 - 2c^2 t (\bar{r} \cdot \bar{v}) + (\bar{r} \cdot \bar{v})^2 + c^2 r^2 - c^4 t^2 - v^2 r^2 + v^2 c^2 t^2$$



$$= (\vec{r} \cdot \vec{v})^2 + (c^2 - v^2)r^2 + c^2(vt)^2 - 2c^2(\vec{r} \cdot \vec{v}t)$$

$$= (\vec{r} \cdot \vec{v})^2 + (c^2 - v^2)r^2 + c^2(r^2 + R^2 - 2\vec{r} \cdot \vec{R}) - 2c^2(r^2 - \vec{r} \cdot \vec{R})$$

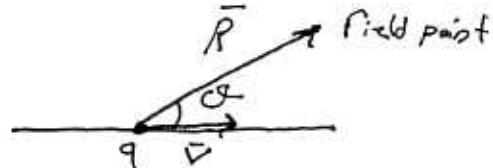
$$\vec{R} = \vec{r} - \vec{v}t$$

$$\vec{v}t = \vec{r} - \vec{R}$$

$$\textcircled{1} \quad = (\vec{r} \cdot \vec{v})^2 - r^2 v^2 + c^2 R^2$$

But, we have

$$\begin{aligned} (\vec{r} \cdot \vec{v})^2 - r^2 v^2 &= ((\vec{R} + \vec{v}t) \cdot \vec{v})^2 - (R + vt)^2 v^2 \\ &= (\vec{R} \cdot \vec{v})^2 + v^4 t^2 + 2(\vec{R} \cdot \vec{v}) v^2 t - R^2 v^2 - 2(\vec{R} \cdot \vec{v}) t v^2 - v^2 t^2 v^2 \\ &= (\vec{R} \cdot \vec{v})^2 - R^2 v^2 = R^2 v^2 \cos^2 \alpha - R^2 v^2 \\ &= -R^2 v^2 (1 - \cos^2 \alpha) \\ &= -R^2 v^2 \sin^2 \alpha \end{aligned}$$



Plugging this result into step ①,

$$S = -R^2 v^2 \sin^2 \alpha + c^2 R^2$$

$$= c^2 R^2 \left(1 - \frac{v^2}{c^2} \sin^2 \alpha \right)$$

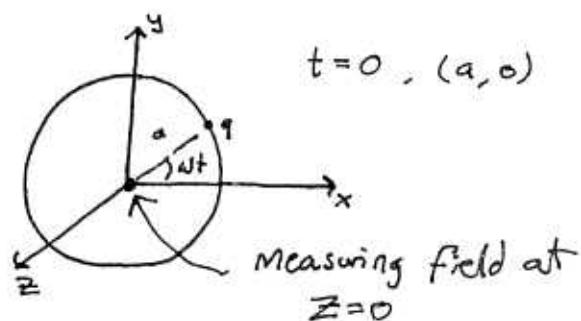
Plugging this into the equation for potential V,

$$V(r, t) = \frac{1}{4\pi\epsilon_0} \frac{q}{R \sqrt{1 - \frac{v^2}{c^2} \sin^2 \alpha}}$$

#7 (Griffiths 10.20)

$$\vec{W}(t) = R(\cos \omega t, \sin \omega t)$$

$$\vec{V}(t) = R\omega(-\sin \omega t, \cos \omega t)$$



$$\bar{a}(t) = -R\omega^2(\cos \omega t, \sin \omega t) = -\omega^2 w(t)$$

$$\bar{r} = \bar{r} - \bar{w}(t_r) = -\bar{w}(t), \quad \bar{u} = c \hat{r} - v(t_r)$$

$$|\bar{r}| = |\bar{w}(t)| = R, \quad t_r = t - R/c$$

$$(eq 10.65) \quad \bar{E}(r, t) = \frac{q}{4\pi\epsilon_0} \frac{r}{(r \cdot \bar{u})^3} (c(c^2 - v^2)\bar{u} + \bar{r} \times (\bar{u} \times \bar{a}))$$

Solving the individual parts of the above equation,

$$\begin{aligned} \bar{u} &= -c(\cos \omega t_r \hat{x} + \sin \omega t_r \hat{y}) - \omega R(-\sin \omega t_r \hat{x} + \cos \omega t_r \hat{y}) \\ &= -(c \cos \omega t_r - \omega R \sin \omega t_r) \hat{x} - (c \sin \omega t_r + \omega R \cos \omega t_r) \hat{y} \end{aligned}$$

$$\bar{r} \times (\bar{u} \times \bar{a}) = (\bar{r} \cdot \bar{a}) \bar{u} - (\bar{r} \cdot \bar{u}) \bar{a}$$

$$\Rightarrow \bar{r} \cdot \bar{a} = -\bar{w} \cdot (-\omega^2 \bar{w}) = \omega^2 R^2$$

$$\begin{aligned} \Rightarrow \bar{r} \cdot \bar{u} &= R(c \cos^2 \omega t_r - \omega R \sin \omega t_r \cos \omega t_r + c \sin^2 \omega t_r + \omega R \sin \omega t_r \cos \omega t_r) \\ &= R c \end{aligned}$$

$$\Rightarrow v^2 = (\omega R)^2$$

Plugging these results into (eq 10.65),

$$\bar{E}(r, t) = \frac{q}{4\pi\epsilon_0} \frac{R}{(Rc)^3} (c(c^2 - v^2)\bar{u} + \omega^2 R^2 \bar{u} - R c (-\omega^2 \bar{w}))$$

$$= \frac{q}{4\pi\epsilon_0} \frac{1}{R^2 c^3} (c^2 - \omega^2 R^2) \bar{u} + \omega^2 R^2 \bar{u} + R c \omega^2 \bar{w}$$

$$\begin{aligned} = \frac{q}{4\pi\epsilon_0} \frac{1}{R^2 c^2} & \left(-c(c \cos \omega t_r - \omega R \sin \omega t_r) \hat{x} - c(c \sin \omega t_r + \omega R \cos \omega t_r) \hat{y} \right. \\ & \left. + R^2 \omega^2 (\cos \omega t_r \hat{x} + \sin \omega t_r \hat{y}) \right) \end{aligned}$$

$$\bar{E} = \frac{q}{4\pi\epsilon_0} \frac{1}{(Rc)^2} \left(((\omega^2 R^2 - c^2) \cos \omega t_r + \omega R c \sin \omega t_r) \hat{x} + ((\omega^2 R^2 - c^2) \sin \omega t_r - \omega R c \cos \omega t_r) \hat{y} \right)$$

$$\begin{aligned}
 (\text{eq 10.66}) \quad \bar{B}(\vec{r}, t) &= \frac{1}{c} \hat{r} \times \bar{E}(r, t) = \frac{1}{c} (\hat{r}_x E_y - \hat{r}_y E_x) \hat{z} \\
 &= -\frac{1}{c} \frac{q}{4\pi\epsilon_0} \frac{1}{(Rc)^2} \left(\cos \omega t_r ((\omega^2 R^2 - c^2) \sin \omega t_r - \omega R c \cos \omega t_r) \right. \\
 &\quad \left. - \sin \omega t_r ((\omega^2 R^2 - c^2) \cos \omega t_r + \omega R c \sin \omega t_r) \right) \hat{z} \\
 &= -\frac{q}{4\pi\epsilon_0} \frac{1}{R^2 c^3} (-\omega R c \cos^2 \omega t_r - \omega R c \sin^2 \omega t_r) \hat{z} \\
 &= \frac{q}{4\pi\epsilon_0} \frac{1}{R^2 c^3} \omega R c \hat{z} = \boxed{\frac{q}{4\pi\epsilon_0} \frac{\omega}{R c^2} \hat{z}}
 \end{aligned}$$

To obtain the field at the center of a ring in terms of current we have

$$q \rightarrow \lambda 2\pi R \quad \text{and} \quad I = \lambda V = \lambda \omega R$$

$$\text{So, } q = \frac{I}{\omega R} 2\pi R = \frac{2\pi I}{\omega}$$

$$\text{Thus, } \bar{B} = \frac{2\pi I}{\omega} \frac{1}{4\pi\epsilon_0} \frac{\omega}{R c^2} \hat{z} = \boxed{\frac{\mu_0 I}{2R} \hat{z}}$$

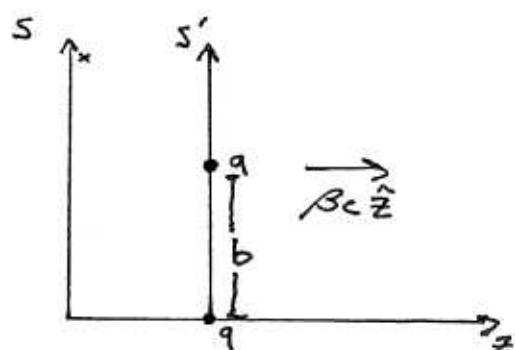
This is the same result as in Exercise 5.6

#8

a) rest frame S' ,

$$F'_x = \frac{q^2}{4\pi\epsilon_0 b^2} \hat{x}$$

S - Lab frame
 S' - Rest frame



b) In the lab frame S,

$$F_x = \frac{dp_x}{dt} = \frac{dp'_x}{\gamma(dt' + v/c dx')}$$

by doing a Lorentz transformation
on the numerator and denominator

$$= \frac{dp'_x}{dt'} \cancel{\gamma(1 + \frac{v}{c^2} \frac{dx'}{dt'})} = F'_x \cancel{\gamma(1 + \frac{v}{c^2} \beta)}$$

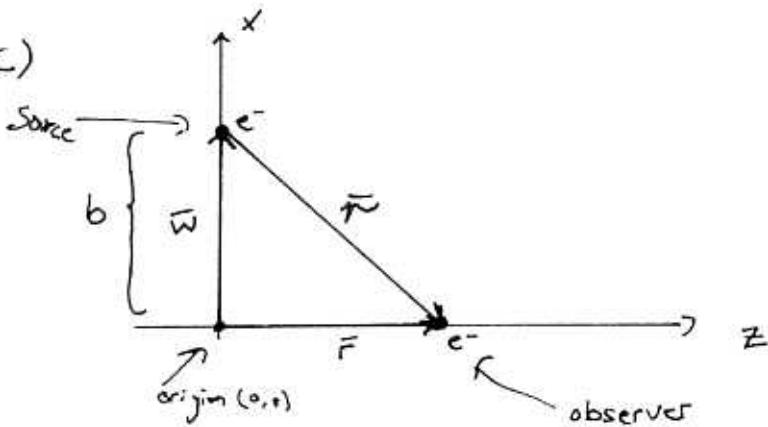
$$= F'_x / \gamma$$

$$\Rightarrow F_x = \frac{q^2}{4\pi\epsilon_0 b^2} \sqrt{1-\beta^2}$$

$$\beta = \frac{v}{c}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

c)



\bar{w} - distance of source from origin

\bar{r} - distance from source to observer

\bar{r}' - distance from origin to observer

From Griffiths we have the following equations,

- $\bar{E}(\bar{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{\bar{r}}{(\bar{r} \cdot \bar{u})^3} ((c^2 - v^2) \bar{u} + \bar{r} \times (\bar{u} \times \bar{a}))$ (eq 10.65)

- $\bar{B}(\bar{r}, t) = \frac{1}{c} \bar{r} \times \bar{E}(\bar{r}, t)$ (eq 10.66)

We will need to know the various quantities in these equations in order to solve them.

$$\bar{w}(t) = b\hat{x} + \beta ct\hat{z}$$

$$\beta = \frac{v}{c}$$

$$\bar{v}(t) = \frac{d\bar{w}}{dt} = \boxed{\beta c \hat{z} = \bar{v}}$$

$$\bar{a}(t) = 0$$

$$\bar{r} = (\nu(\infty) + \beta c t) \hat{z} = \beta(r + ct) \hat{z}$$

$$\bar{r} = \bar{r} - \bar{\omega} = \beta(r + ct) \hat{z} - b\hat{x} - \beta ct \hat{z} = \boxed{\beta r \hat{z} - b\hat{x} = \bar{v}}$$

$$|\bar{v}| = \sqrt{\beta^2 r^2 + b^2}$$

$$\Rightarrow v^2 = \beta^2 r^2 + b^2 \Rightarrow v^2(1-\beta^2) = b^2$$

$$\Rightarrow \boxed{v = (1-\beta^2)^{-1/2} b}$$

$$\begin{aligned} \bar{u} &= c\bar{v} - \bar{v} = c\bar{v}/v - \bar{v} = c(\beta\hat{z} - \frac{b}{\bar{v}}\hat{x} - \beta\hat{z}) \\ &= -\frac{cb}{v}\hat{x} = -\frac{c\cancel{b}}{(1-\beta^2)^{1/2}b}\hat{x} = -c(1-\beta^2)^{1/2}\hat{x} \end{aligned}$$

$$\Rightarrow \boxed{\bar{u} = -c(1-\beta^2)^{1/2}\hat{x}}$$

Now, we begin to solve the individual pieces of (eq 10.65),

$$\bar{v} \cdot \bar{u} = (1-\beta^2)^{1/2}cb$$

$$\bar{r} \times (\bar{u} \times \vec{a}) = 0$$

Now, we can put all these results into the equation for the Electric field (eq 10.65):

$$\bar{E}(\bar{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{r}{(\bar{v} \cdot \bar{u})^2} ((c^2 - v^2)\bar{u} + \bar{r} \times (\bar{u} \times \vec{a}))$$

$$= \frac{q}{4\pi\epsilon_0} \frac{(1-\beta^2)^{-1/2}b}{(1-\beta^2)^{3/2}c^3 b^2} ((c^2 - \beta^2 c^2)(1-\beta^2)^{1/2}\hat{x})$$

$$= \frac{q}{4\pi\epsilon_0} \frac{1}{(1-\beta^2)^2 c^3 b^2} (-c^3 (1-\beta^2)^{3/2}\hat{x})$$

⑤

$$= -\frac{q}{4\pi\epsilon_0} \frac{1}{(1-\beta^2)^{1/2} b^2} \hat{x}$$

$$\Rightarrow \boxed{\bar{E} = -\frac{1}{4\pi\epsilon_0} \frac{q}{(1-\beta^2)^{1/2} b^2} \hat{x}}$$

Now, we solve for the total force using (eq 10.67),

$$\begin{aligned}\bar{F} &= q(\bar{E} + \bar{v} \times \bar{B}) \\ &= q(\bar{E} + \bar{v} \times (\frac{1}{c} \hat{r} \times \bar{E})) \\ &= -\frac{1}{4\pi\epsilon_0} \frac{q^2}{(1-\beta^2)^{1/2} b^2} \left(\hat{x} + \beta c \hat{z} \times \left(\frac{1}{c} (\beta \hat{z} - \frac{b}{r} \hat{x}) \times \hat{x} \right) \right) \\ &= -\frac{1}{4\pi\epsilon_0} \frac{q^2}{(1-\beta^2)^{1/2} b^2} (\hat{x} + \beta^2 \hat{z} \times \hat{y}) \\ &= -\frac{1}{4\pi\epsilon_0} \frac{q^2}{(1-\beta^2)^{1/2} b^2} (1-\beta^2) \hat{x} \\ \Rightarrow \boxed{\bar{F}_x = -\frac{1}{4\pi\epsilon_0} \frac{q^2}{b^2} \sqrt{1-\beta^2} \hat{x}}\end{aligned}$$

• Which is the same result that we got in part a)